

# Vertex Operator of $U_q(\widehat{B}_l)$ for Level One

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## Abstract

In this paper, we give the explicit formulae of vertex operators of  $U_q(\widehat{B}_l)$  for level-one as operators on the Fock space. Meanwhile, we point out that the free field realization (by one fermionic field and  $l$  bosonic fields) of highest weight module with highest weight  $\Lambda_l$  has two irreducible modules.

## 1 Introduction

One of the central subjects of mathematical physics has been the studies on exactly solvable models in two dimensions for many years. The central problem is eigenstate and correlation functions (form-factors) in exactly solvable models. In [3] [7] a new scheme was given for solving the six-vertex model and associated XXZ chain in the antiferromagnetic regime using the newly discovered quantum affine symmetry of the system. The approach of that paper has been extended to higher spin-chains [8] [9] [10] [11], to the higher rank case [1] [12] and to the ABF models [14]. The analogous approach in integrable massive field theory also has been developed by Lukyanov [15]. All of these papers are concerned with models constructed on the quantum affine algebra  $U_q(\widehat{A}_l)$ . The key object in this approach is vertex operators which have first been introduced by Frenkel and Reshetikhin [2]. In order to extend that scheme to the models with symmetry of  $U_q(\widehat{B}_l)$ , we construct the  $q$ -vertex operators related to  $U_q(\widehat{B}_l)$  in this paper. The physical application will be

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developed else where. In section 2, we briefly recall the free field realization of  $U_q(\widehat{B}_l)$ . In section 3, we deduce the vertex operators.

## 2 Free Field Realization of $U_q(\widehat{B}_l)$

In this section we briefly recall the free field realization of  $U_q(\widehat{B}_l)$  [4]

### 2.1 Definition of $U_q(\widehat{B}_l)$ .

First, fix some notations we will use. Let  $\widehat{P} = \mathbf{Z}\Lambda_0 \oplus \mathbf{Z}\Lambda_1 \oplus \cdots \oplus \mathbf{Z}\Lambda_l \oplus \mathbf{Z}\delta$  be the weight lattice of  $U_q(\widehat{B}_l)$  and let  $\alpha_i = \sum_{j=0}^l a_{ji} \Lambda_j + \delta_{i,0} \delta$  ( $i = 0, 1, \dots, l$ ) be simple roots. So

we have  $\delta = \alpha_0 + \alpha_1 + 2(\alpha_2 + \cdots + \alpha_l)$  and  $\widehat{Q} = \mathbf{Z}\alpha_0 \oplus \mathbf{Z}\alpha_1 \oplus \cdots \oplus \mathbf{Z}\alpha_l$  being its root lattice. The symmetric bilinear form  $(,)$  on  $\widehat{P}$  is defined by

$$(\Lambda_0, \Lambda_i) = 0 \quad i = 0, 1, \dots, l, \quad (\Lambda_0, \delta) = 1$$

$$(\Lambda_i, \alpha_j) = d_i \delta_{ij}$$

we can get

$$(\Lambda_1, \delta) = 1, \quad (\delta, \delta) = 0, \quad (\Lambda_l, \delta) = 1$$

$$(\Lambda_i, \Lambda_j) = d_i (\overline{A}^{-1})_{ij} \quad 1 \leq i, j \leq l, \quad (\Lambda_i, \delta) = 2 \quad 1 < i < l$$

$$(\Lambda_i, \delta) = 2 \quad 1 \leq i \leq l, \quad (\alpha_i, \alpha_j) = d_i a_{ij}$$

where  $A = (a_{ij})_{i,j=0}^l$  is the Cartan matrix of  $U_q(\widehat{B}_l)$  and  $\overline{A}$  is the matrix of  $A$  removed by first line and first column.

For  $U_q(\widehat{B}_2)$ ,

$$d_0 = d_1 = 2d_2 = 1,$$

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -2 & -2 & 2 \end{bmatrix}$$

and for  $U_q(\widehat{B}_l)$  ( $l \geq 3$ ),

$$d_0 = d_1 = \cdots = d_{l-1} = 2d_l = 1,$$

$$A = \begin{bmatrix} 2 & 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 2 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdots & -2 & 2 \end{bmatrix}$$

Define the dual space of  $\widehat{P}$  as  $\widehat{P}^* = \mathbf{Z}h_0 \oplus \mathbf{Z}h_1 \oplus \cdots \oplus \mathbf{Z}h_l \oplus \mathbf{Z}d$ . The dual pairing  $\langle , \rangle$  is defined by

$$\begin{aligned}\langle h_i, \lambda \rangle &:= d_i^{-1}(\alpha_i, \lambda) \quad \lambda \in \widehat{P} \\ \langle d, \lambda \rangle &:= (\Lambda_0, \lambda) \quad \lambda \in \widehat{P}\end{aligned}$$

Later we will use the weight lattice and the root lattice of  $U_q(B_l)$ . The weight lattice  $P = \mathbf{Z}\lambda_1 \oplus \mathbf{Z}\lambda_2 \oplus \cdots \oplus \mathbf{Z}\lambda_l$ ,  $\lambda_i = \Lambda_i - a_i^\vee \Lambda_0$ , where  $a_0^\vee = a_1^\vee = a_l^\vee = 1, a_2^\vee = \cdots = a_{l-1}^\vee = 2$ . The root lattice of  $U_q(B_l)$  is  $Q = \mathbf{Z}\alpha_1 \oplus \mathbf{Z}\alpha_2 \oplus \cdots \oplus \mathbf{Z}\alpha_l$ . In this article, we assume  $-1 < q < 0$ . and use the following standard notations:

$$\begin{aligned}\omega &= \frac{1}{(q^{\frac{1}{2}} + q^{-\frac{1}{2}})}, \quad q_i = q^{d_i} \\ [m]_{q_i} &= \frac{q_i^m - q_i^{-m}}{q_i - q_i^{-1}} \quad m \in \mathbf{Z} \\ \left[ \begin{array}{c} n \\ m \end{array} \right]_{q_i} &= \frac{[1]_{q_i} [2]_{q_i} \cdots [n]_{q_i}}{[1]_{q_i} [2]_{q_i} \cdots [m]_{q_i} [1]_{q_i} [2]_{q_i} \cdots [n-m]_{q_i}}\end{aligned}$$

and when  $q_i = q$ , we omit the index. Quantum affine algebra  $U_q(\widehat{B}_l)$  is an associative algebra over  $\mathcal{C}$  with unity generated by  $e_i, f_i, q_i^{\pm h_i}$  ( $i = 0, 1, 2, \dots, l$ ) and  $q^{\pm d}$ .  $U'_q(\widehat{B}_l)$  is the subalgebra of  $U_q(\widehat{B}_l)$  generated by  $\{q_i^{\pm h_i}, e_i, f_i \mid 0 \leq i \leq l\}$ . The defining relations are as follows ( $\forall h, h' \in \widehat{P}^*$ ):

$$\begin{aligned}q^h q^{h'} &= q^{h+h'}, \quad q^0 = 1 \\ q^h e_i q^{-h} &= q^{\langle h, \alpha_i \rangle} e_i \\ q^h f_i q^{-h} &= q^{-\langle h, \alpha_i \rangle} f_i \\ [e_i, f_j] &= \delta_{i,j} \frac{q_i^{h_i} - q_i^{-h_i}}{q_i - q_i^{-1}} \\ \sum_{n=0}^{1-a_{ij}} (-1)^n \left[ \begin{array}{c} 1-a_{ij} \\ n \end{array} \right]_{q_i} e_i^{1-a_{ij}-n} e_j e_i^n &= 0, \quad i \neq j \\ \sum_{n=0}^{1-a_{ij}} (-1)^n \left[ \begin{array}{c} 1-a_{ij} \\ n \end{array} \right]_{q_i} f_i^{1-a_{ij}-n} f_j f_i^n &= 0, \quad i \neq j\end{aligned}$$

The algebra  $U_q(\widehat{B}_l)$  has a Hopf algebra structure with the following coproduct  $\Delta : U_q(\widehat{B}_l) \rightarrow U_q(\widehat{B}_l) \otimes U_q(\widehat{B}_l)$

$$\Delta(e_i) = e_i \otimes 1 + q_i^{h_i} \otimes e_i, \quad \Delta(f_i) = f_i \otimes q_i^{-h_i} + 1 \otimes f_i$$

$$\Delta(q^h) = q^h \otimes q^h, \quad h \in P^*$$

As above this algebra is defined through Chevalley generators.

## 2.2 Drinfeld realization [6]

For free field realization,  $U_q(\widehat{B}_l)$  defined by Chevalley generators is not convenient. Drinfeld has given another realization in 1988 [6]. The new realization is an associative algebra generated by the elements  $\{x_i^\pm(n), a_i(m), \gamma^{\pm\frac{1}{2}}, q_i^{h_i} \mid 1 \leq i \leq l, m \in \mathbf{Z} \setminus \{0\}, n \in \mathbf{Z}\}$  satisfying the follow relations:

$\gamma^{\pm\frac{1}{2}}$  is the center of the algebra.

$$[a_i(n), a_j(m)] = \delta_{n+m,0} \frac{1}{n} [na_{ij}]_{q_i} \frac{\gamma^n - \gamma^{-n}}{q_j - q_j^{-1}}$$

$$q_i^{h_i} x_j^\pm(n) q_i^{-h_i} = q_i^{\pm a_{ij}} x_j^\pm(n)$$

$$[a_i(n), x_j^\pm(m)] = \pm \frac{1}{n} [na_{ij}]_{q_i} \gamma^{\mp \frac{|n|}{2}} x_j^\pm(n+m)$$

$$x_i^\pm(n+1) x_j^\pm(m) - q_i^{\pm a_{ij}} x_j^\pm(m) x_i^\pm(n+1) = q_i^{\pm a_{ij}} x_i^\pm(n) x_j^\pm(m+1) - x_j^\pm(m+1) x_i^\pm(n)$$

$$[x_i^+(n), x_j^-(m)] = \delta_{i,j} \frac{\gamma^{\frac{n-m}{2}} \varphi_i^+(n+m) - \gamma^{-\frac{n-m}{2}} \varphi_i^-(n+m)}{q_i - q_i^{-1}}$$

$$\sum_{\pi \in \sum_p} \sum_{k=0}^p (-1)^k \left[ \begin{array}{c} p \\ k \end{array} \right]_{q_i} x_i^\pm(r_{\pi(1)}) \cdots x_i^\pm(r_{\pi(k)}) x_j^\pm(s) x_i^\pm(r_{\pi(k+1)}) \cdots x_j^\pm(r_{\pi(p)}) = 0$$

if  $i \neq j$ , for sequences of integers  $r_1, \dots, r_p$ . where  $p = 1 - a_{ij}$ ,  $\sum_p$  is the symmetric group on  $p$  letters, and the  $\varphi_i^\pm(r)$  are determined by equating powers of  $u$  in the formal power series

$$\sum_{r=0}^{\infty} \varphi_i^\pm(r) u^{\mp r} = q_i^{\pm h_i} \exp(\pm(q_i - q_i^{-1})) \sum_{n=1}^{\infty} a_i(\pm n) u^{\mp n}$$

Define maps  $w_i^\pm : U_q(\widehat{g}) \rightarrow U_q(\widehat{g})$  by

$$w_i^\pm a = x_i^\pm(0) a - q_i^{\pm h_i} a q_i^{\mp h_i} x_i^\pm(0)$$

then, the isomorphism between the two realization is

$$q^{h_0} = \gamma q^{-h_1-2h_2-\cdots-2h_{l-1}-h_l}, \quad e_i = x_i^+(0), \quad f_i = x_i^-(0), \quad \text{for } i = 1, 2, \dots, l$$

$$e_0 = w^-_2 w^-_3 \cdots w^-_l w^-_l \cdots w^-_2 X_1^-(-1) q^{h_0} \gamma^{-1} = e'_0 q^{h_0} \gamma^{-1}$$

$$f_0 = q^{2l-3} \omega^2 q^{-h_0} \gamma w^+_2 w^+_3 \cdots w^+_l w^+_l \cdots w^+_2 X_1^+(-1) = q^{-h_0} \gamma f'_0$$

### 2.3 Central Extension of Weight Lattice

It is well known that it is enough to consider only the central extension of root lattice  $Q$  to construct the representation of  $U_q(\widehat{B}_l)$ . But, for the construction of the vertex operators it is necessary to define the central extention of weight lattice  $P$ .

Define the group algebra  $C[\overline{P}]$  generated by the symbols  $\{e^{\lambda_1}, e^{\alpha_1}, e^{\alpha_2}, \dots, e^{\alpha_{l-1}}\}$  and satisfying the following relations:

$$e^{\alpha_i} e^{\alpha_j} = (-1)^{(\alpha_i, \alpha_j)} e^{\alpha_j} e^{\alpha_i} \quad 1 \leq i, j \leq l-1$$

$$e^{\alpha_i} e^{\lambda_1} = (-1)^{\delta_{i,1}} e^{\lambda_1} e^{\alpha_i} \quad 1 \leq i \leq l-1$$

For  $\alpha = m_0 \lambda_1 + m_1 \alpha_1 + m_2 \alpha_2 + \cdots + m_{l-1} \alpha_{l-1}$ , we denote  $e^\alpha = e^{m_0 \lambda_1} e^{m_1 \alpha_1} e^{m_2 \alpha_2} \cdots e^{m_{l-1} \alpha_{l-1}}$ . So a simple calculation shows

$$e^{\alpha_l} e^{\lambda_1} = -e^{\lambda_1} e^{\alpha_l}$$

$$e^{\alpha_i} e^{\alpha_j} = (-1)^{(\alpha_i, \alpha_j)} e^{\alpha_j} e^{\alpha_i} \quad 1 \leq i, j \leq l$$

### 2.4 Level One Module

Bernard [4] has given the free field realization of level one module by using one fermionic field and  $l$  bosonic fields. Here, we reconstruct the three level one module and point out a fact having been overlooked.

Let

$$H_i := C[a_j(-m), \Psi(-k) (1 \leq j \leq l, m \in \mathbf{Z}_{\geq 0}, k \in \mathbf{Z}_{\geq 0} + \frac{1}{2})] \otimes C[\overline{Q}] e^{\Lambda_i}$$

for  $i = 0, 1$ , in  $\mathcal{NS}$  cases.

$$H_l := C[a_j(-m), \Psi(-k) (1 \leq j \leq l, m \in \mathbf{Z}_{\geq 0}, k \in \mathbf{Z}_{\geq 0})] \otimes C[\overline{Q}] e^{\Lambda_i}$$

in  $\mathcal{R}$  cases.

We define the operators  $a_i(k)$  ( $1 \leq i \leq l$ ),  $\Psi(k)$ ,  $\partial_{\alpha_i}$ ,  $e^{\alpha_i}$ ,  $d$  on  $H_i$  ( $i = 0, 1, l$ ) as follows: for  $f \otimes e^\beta = a_{i_1}(-n_1) \cdots a_{i_k}(-n_k) \Psi(-k_1) \cdots \Psi(-k_n) \otimes e^\beta \in H_i$ ,

$$a_j(k) \cdot f \otimes e^\beta = \begin{cases} a_j(k) f \otimes e^\beta & (k < 0) \\ [a_j(k), f] \otimes e^\beta & (k > 0) \end{cases}$$

$$\begin{aligned}
\Psi(k).f \otimes e^\beta &= \begin{cases} \Psi(k)f \otimes e^\beta & (k \leq 0) \\ \{\Psi(k), f\} \otimes e^\beta & (k > 0) \end{cases} \\
\partial_{\alpha_i}.f \otimes e^\beta &= (\alpha_i, \beta)f \otimes e^\beta \\
e^{\alpha_j}.f \otimes e^\beta &= f \otimes e^{\alpha_j}e^\beta \\
d.f \otimes e^\beta &= \left( -\sum_{l=1}^k n_l - \sum_{l=1}^n \left( k_l - \frac{1}{2} \right) - \frac{(\beta, \beta)}{2} + \frac{(\Lambda_i, \Lambda_i)}{2} \right) f \otimes e^\beta
\end{aligned}$$

Let

$$x_i^\pm(z) := \sum_{n \in \mathbf{Z}} x_i^\pm(n) z^{-n} \quad (1 \leq i \leq l-1)$$

We define the action of  $U_q(\widehat{B}_l)$ :

$$\begin{aligned}
\gamma &\rightarrow q, \quad q_j^{h_j} \rightarrow q^{\partial_{\alpha_j}}, \quad (1 \leq j \leq l) \\
x_i^\pm(z) &\rightarrow z \exp\left(\pm \sum_{k=1}^{\infty} \frac{a_i(-k)}{[k]} q^{\mp \frac{k}{2}} z^k\right) \exp\left(\mp \sum_{k=1}^{\infty} \frac{a_i(k)}{[k]} q^{\mp \frac{k}{2}} z^{-k}\right) e^{\pm \alpha_i} z^{\pm \partial_{\alpha_i}} \\
x_l^\pm(z) &\rightarrow z^{\frac{1}{2}} \exp\left(\pm \sum_{k=1}^{\infty} \frac{a_l(-k)}{[k]} q^{\mp \frac{k}{2}} \omega z^k\right) \exp\left(\mp \sum_{k=1}^{\infty} \frac{a_l(k)}{[k]} q^{\mp \frac{k}{2}} \omega z^{-k}\right) e^{\pm \alpha_l} z^{\pm \partial_{\alpha_l}} \Psi(z)
\end{aligned}$$

where

$$\Psi(z) = \sum_n \Psi(n) z^{-n}$$

with  $n \in \mathbf{Z} + \frac{1}{2}$  (or  $n \in \mathbf{Z}$ ) in the  $\mathcal{NS}$  (or  $\mathcal{R}$ ) cases, respectively, and

$$\{\Psi(n), \Psi(m)\} = (q^n + q^{-n}) \delta_{n+m,0}$$

$$[a_i(n), a_j(m)] = \delta_{n+m,0} \frac{1}{n} [na_{ij}]_{q_i} \frac{q^n - q^{-n}}{q_j - q_j^{-1}}$$

$$[a_i(n), \Psi(m)] = 0$$

Define

$$G = (-1)^{2 \sum_{i=1}^l \partial_{\lambda_i}} + N_F$$

for  $\mathcal{NS}$  cases.

$$G = (-1)^{2 \sum_{i=1}^l \partial_{\lambda_i}} + N_F - 2 \sum_{i=1}^l (\lambda_i, \lambda_l)$$

for  $\mathcal{R}$  cases. where  $N_F$  denotes the fermion's number operator. We easily find  $G$  commutes with all the elements of  $U_q(\widehat{B}_l)$ . Through the eigenvalues of  $G$ , we can divide Fock space into irreducible ones. Four irreducible  $U_q(\widehat{B}_l)$ -modules  $V(\Lambda_0), V(\Lambda_1), V(\Lambda_l), V(\Lambda'_l)$  whose highest weight vectors are  $1 \otimes e^{\Lambda_0}, 1 \otimes e^{\Lambda_1}, 1 \otimes e^{\Lambda_l}, \Psi(0) \otimes e^{\Lambda'_l}$ , respectively. The first two modules are in  $\mathcal{NS}$  cases and the others are in  $\mathcal{R}$  cases. We have not seen that the reducibility in  $\mathcal{R}$  cases has been discussed before.

### 3. Vertex Operators

Let us review the definition and some properties of the vertex operators.

#### 3.1 Finite dimensional $U'_q(\widehat{B}_l)$ module

Let  $V$  be a finite dimensional  $U'_q(\widehat{B}_l)$ -module with basis  $\{v_m \mid 1 \leq m \leq 2l+1\}$ , and the representation of  $e_i, f_i, h_i$  is as follows:

$$e_i = f_i^t = E_{i,i+1} + E_{2l-i+1,2l-i+2}, \quad i \neq 0, l$$

$$e_l = f_l^t = \omega^{-\frac{1}{2}}(E_{l,l+1} + E_{l+1,l+2})$$

$$e_0 = f_0^t = E_{2l,1} + E_{2l+1,2}$$

$$h_i = E_{i,i} - E_{i+1,i+1} + E_{2l-i+1,2l-i+1} - E_{2l-i+2,2l-i+2}, \quad i \neq 0, l$$

$$h_0 = -E_{1,1} - E_{2,2} + E_{2l,2l} + E_{2l+1,2l+1}$$

$$h_l = 2E_{l,l} - 2E_{l+2,l+2}$$

Define the  $U_q$ -module structure on  $V_z$  as follows.

$$e_i(v_m \otimes z^n) = e_i v_m \otimes z^{n+\delta_{i,0}}, \quad f_i(v_m \otimes z^n) = f_i v_m \otimes z^{n-\delta_{i,0}}$$

$$h_i(v_m \otimes z^n) = h_i v_m \otimes z^n, \quad d(v_m \otimes z^n) = n v_m \otimes z^n$$

We call  $V_z$  the affinization of  $V$  as a  $U_q(\widehat{B}_l)$ -module of level zero.

#### 3.2 Definition of $q$ -Vertex Operators

The intertwiners of  $U_q(\widehat{B}_l)$ -modules

$$\widehat{\Psi}_{\Lambda_i}^{\Lambda_j V}(z) : V(\Lambda_i) \rightarrow V(\Lambda_j) \otimes V_z$$

are called type-I vertex operators and the operators

$$\widehat{\Psi}_{\Lambda_i}^{V \Lambda_j}(z) : V(\Lambda_i) \rightarrow V_z \otimes V(\Lambda_j)$$

are called type-II vertex operators, where  $\otimes$  is the tensor product with an appropriate completion. Denote the vertex operators as a formal series

$$\begin{aligned}\widehat{\Psi}_{\Lambda_i}^{\Lambda_j V}(z) &= \sum_{m=1}^{2l+1} \widehat{\Psi}_{\Lambda_i m}^{\Lambda_j V}(z) \otimes v_m \\ \widehat{\Psi}_{\Lambda_i}^{V \Lambda_j}(z) &= \sum_{m=1}^{2l+1} v_m \otimes \widehat{\Psi}_{\Lambda_i m}^{V \Lambda_j}(z) \\ \widehat{\Psi}_{\Lambda_i m}^{\Lambda_j V}(z) &= \sum_{n \in \mathbf{Z}} \widehat{\Psi}_{\Lambda_i m}^{\Lambda_j V}(n) z^n \\ \widehat{\Psi}_{\Lambda_i m}^{V \Lambda_j}(z) &= \sum_{n \in \mathbf{Z}} \widehat{\Psi}_{\Lambda_i m}^{V \Lambda_j}(n) z^n\end{aligned}$$

There exist four type-I (respectively type-II) vertex operators [2] [13].

$$\begin{aligned}\widehat{\Psi}_{\Lambda_0}^{\Lambda_1 V}(z) &: V(\Lambda_0) \rightarrow V(\Lambda_1) \otimes V_z \\ \widehat{\Psi}_{\Lambda_1}^{\Lambda_0 V}(z) &: V(\Lambda_1) \rightarrow V(\Lambda_0) \otimes V_z \\ \widehat{\Psi}_{\Lambda_l}^{\Lambda_l' V}(z) &: V(\Lambda_l) \rightarrow V(\Lambda_l') \otimes V_z \\ \widehat{\Psi}_{\Lambda_l'}^{\Lambda_l V}(z) &: V(\Lambda_l') \rightarrow V(\Lambda_l) \otimes V_z\end{aligned}$$

We can impose the normalization condition.

$$\begin{aligned}\widehat{\Psi}_{\Lambda_0}^{\Lambda_1 V}(z) \cdot (1 \otimes e^{\Lambda_0}) &= (1 \otimes e^{\Lambda_1}) \otimes v_{2l+1} + (\text{terms of positive powers in } z) \\ \widehat{\Psi}_{\Lambda_1}^{\Lambda_0 V}(z) \cdot (1 \otimes e^{\Lambda_1}) &= (1 \otimes e^{\Lambda_0}) \otimes v_1 + (\text{terms of positive powers in } z) \\ \widehat{\Psi}_{\Lambda_l}^{\Lambda_l' V}(z) \cdot (1 \otimes e^{\Lambda_l}) &= (\Psi(0) \otimes e^{\Lambda_l}) \otimes v_{l+1} + (\text{terms of positive powers in } z) \\ \widehat{\Psi}_{\Lambda_l'}^{\Lambda_l V}(z) \cdot (\Psi(0) \otimes e^{\Lambda_l}) &= (1 \otimes e^{\Lambda_l}) \otimes v_{l+1} + (\text{terms of positive powers in } z)\end{aligned}$$

Later we will find the formulae of  $\widehat{\Psi}_{\Lambda_l}^{\Lambda_l' V}(z)$  and  $\widehat{\Psi}_{\Lambda_l'}^{\Lambda_l V}(z)$  identical. So we will denote them as  $\widehat{\Psi}_{\Lambda_l}^{\Lambda_l V}(z)$ .

### 3.3 Vertex Operator of Type-I

From the intertwining relation (we only need to describe the type-I vertex operator, the type-II is quite parallel).

$$\Delta(x) \circ \widehat{\Psi}_{\Lambda_i}^{\Lambda_j V}(z) = \widehat{\Psi}_{\Lambda_i}^{\Lambda_j V}(z) \circ x$$

then we can get the commutation relations between the type-I vertex operator with Chevalley generators. Here we write the relations partially.

$$q f_2 \widehat{\Psi}_{\Lambda_{\sigma_1} 3}^{\Lambda_{\sigma_2} V}(z) + \widehat{\Psi}_{\Lambda_{\sigma_1} 2}^{\Lambda_{\sigma_2} V}(z) = \widehat{\Psi}_{\Lambda_{\sigma_1} 3}^{\Lambda_{\sigma_2} V}(z) f_2$$

$$\begin{aligned}
qf_3\widehat{\Psi}_{\Lambda_{\sigma_1}4}^{\Lambda_{\sigma_2}V}(z) + \widehat{\Psi}_{\Lambda_{\sigma_1}3}^{\Lambda_{\sigma_2}V}(z) &= \widehat{\Psi}_{\Lambda_{\sigma_1}4}^{\Lambda_{\sigma_2}V}(z)f_3 \\
&\vdots \quad \vdots \quad \vdots \\
qf_{l-1}\widehat{\Psi}_{\Lambda_{\sigma_1}l}^{\Lambda_{\sigma_2}V}(z) + \widehat{\Psi}_{\Lambda_{\sigma_1}l-1}^{\Lambda_{\sigma_2}V}(z) &= \widehat{\Psi}_{\Lambda_{\sigma_1}l}^{\Lambda_{\sigma_2}V}(z)f_{l-1} \\
f_l\widehat{\Psi}_{\Lambda_{\sigma_1}l+1}^{\Lambda_{\sigma_2}V}(z) + \omega^{-\frac{1}{2}}\widehat{\Psi}_{\Lambda_{\sigma_1}l}^{\Lambda_{\sigma_2}V}(z) &= \widehat{\Psi}_{\Lambda_{\sigma_1}l+1}^{\Lambda_{\sigma_2}V}(z)f_l \\
qf_l\widehat{\Psi}_{\Lambda_{\sigma_1}l+2}^{\Lambda_{\sigma_2}V}(z) + \omega^{-\frac{1}{2}}\widehat{\Psi}_{\Lambda_{\sigma_1}l+1}^{\Lambda_{\sigma_2}V}(z) &= \widehat{\Psi}_{\Lambda_{\sigma_1}l+2}^{\Lambda_{\sigma_2}V}(z)f_l \\
qf_{l-1}\widehat{\Psi}_{\Lambda_{\sigma_1}l+3}^{\Lambda_{\sigma_2}V}(z) + \widehat{\Psi}_{\Lambda_{\sigma_1}l+2}^{\Lambda_{\sigma_2}V}(z) &= \widehat{\Psi}_{\Lambda_{\sigma_1}l+3}^{\Lambda_{\sigma_2}V}(z)f_{l-1} \\
&\vdots \quad \vdots \quad \vdots \\
qf_2\widehat{\Psi}_{\Lambda_{\sigma_1}2l}^{\Lambda_{\sigma_2}V}(z) + \widehat{\Psi}_{\Lambda_{\sigma_1}2l-1}^{\Lambda_{\sigma_2}V}(z) &= \widehat{\Psi}_{\Lambda_{\sigma_1}2l}^{\Lambda_{\sigma_2}V}(z)f_2 \\
qf_1\widehat{\Psi}_{\Lambda_{\sigma_1}2l+1}^{\Lambda_{\sigma_2}V}(z) + \widehat{\Psi}_{\Lambda_{\sigma_1}2l}^{\Lambda_{\sigma_2}V}(z) &= \widehat{\Psi}_{\Lambda_{\sigma_1}2l+1}^{\Lambda_{\sigma_2}V}(z)f_1 \\
q^{-1}e'_0\widehat{\Psi}_{\Lambda_{\sigma_1}2l+1}^{\Lambda_{\sigma_2}V}(z) + q^2z\widehat{\Psi}_{\Lambda_{\sigma_1}2l}^{\Lambda_{\sigma_2}V}(z) &= \widehat{\Psi}_{\Lambda_{\sigma_1}2l+1}^{\Lambda_{\sigma_2}V}(z)e'_0 \\
[e_i, \widehat{\Psi}_{\Lambda_{\sigma_1}2l+1}^{\Lambda_{\sigma_2}V}(z)] &= 0, \quad i = 1, \dots, l \\
&\quad [e_i, \widehat{\Psi}_{\Lambda_{\sigma_1}i+1}^{\Lambda_{\sigma_2}V}(z)] = 0, \quad i \neq l \\
&\quad [e_i, \widehat{\Psi}_{\Lambda_{\sigma_1}2l-i+1}^{\Lambda_{\sigma_2}V}(z)] = 0, \quad i \neq l \\
q^{h_i}\widehat{\Psi}_{\Lambda_{\sigma_1}i+1}^{\Lambda_{\sigma_2}V}(z)q^{-h_i} &= q\widehat{\Psi}_{\Lambda_{\sigma_1}i+1}^{\Lambda_{\sigma_2}V}(z), \quad i \neq l \\
q^{h_i}\widehat{\Psi}_{\Lambda_{\sigma_1}2l-i+2}^{\Lambda_{\sigma_2}V}(z)q^{-h_i} &= q\widehat{\Psi}_{\Lambda_{\sigma_1}2l-i+2}^{\Lambda_{\sigma_2}V}(z), \quad i \neq l \\
&\quad [e_l, \widehat{\Psi}_{\Lambda_{\sigma_1}l+2}^{\Lambda_{\sigma_2}V}(z)] = 0, \quad [q_l^{h_l}, \widehat{\Psi}_{\Lambda_{\sigma_1}l+1}^{\Lambda_{\sigma_2}V}(z)] = 0, \quad q_l^{h_l}\widehat{\Psi}_{\Lambda_{\sigma_1}l+2}^{\Lambda_{\sigma_2}V}(z)q_l^{-h_l} = q\widehat{\Psi}_{\Lambda_{\sigma_1}l+2}^{\Lambda_{\sigma_2}V}(z) \\
&\quad q^{h_0}\widehat{\Psi}_{\Lambda_{\sigma_1}2l+1}^{\Lambda_{\sigma_2}V}(z)q^{-h_0} = q^{-1}\widehat{\Psi}_{\Lambda_{\sigma_1}2l+1}^{\Lambda_{\sigma_2}V}(z)
\end{aligned}$$

Define operators  $P^\pm_i$

$$P^\pm_i x = [x^\pm_i(0), x]q^{\mp h_i}$$

we have

$$P^\pm_i W_j^\mp x = \frac{(q^{\pm h_j}xq^{\mp h_j} - q^{\mp h_j}xq^{\pm h_j})}{q_i - q_i^{-1}}\delta_{i,j} + W_j^\mp P^\pm_i x$$

Using them, we get

$$P^+_2 P^+_3 \cdots P^+_l P^+_l \cdots P^+_3 P^+_2 e'_0 = (q^{\frac{1}{2}} + q^{-\frac{1}{2}})^2 x^-_1(1)$$

and the consistent identity

$$q^{2l-2} \omega(\widehat{\Psi}_{\Lambda_{\sigma_1} 2l+1}^{\Lambda_{\sigma_2} V}(z) f_1 - q f_1 \widehat{\Psi}_{\Lambda_{\sigma_1} 2l+1}^{\Lambda_{\sigma_2} V}(z)) = (qz)^{-1} (\widehat{\Psi}_{\Lambda_{\sigma_1} 2l+1}^{\Lambda_{\sigma_2} V}(z) x^-_1(1) - q^{-1} x^-_1(1) \widehat{\Psi}_{\Lambda_{\sigma_1} 2l+1}^{\Lambda_{\sigma_2} V}(z))$$

Commuting above identity with  $X_1^+(w)$ , we get

$$\begin{aligned} & \frac{q^{2l} z \omega}{w} (\widehat{\Psi}_{\Lambda_{\sigma_1} 2l+1}^{\Lambda_{\sigma_2} V}(z) \widehat{\varphi}_1^+(w \gamma^{-\frac{1}{2}}) - q^2 \widehat{\varphi}_1^+(w \gamma^{-\frac{1}{2}}) \widehat{\Psi}_{\Lambda_{\sigma_1} 2l+1}^{\Lambda_{\sigma_2} V}(z)) \\ &= \widehat{\Psi}_{\Lambda_{\sigma_1} 2l+1}^{\Lambda_{\sigma_2} V}(z) \widehat{\varphi}_1^+(w \gamma^{-\frac{1}{2}}) - \widehat{\varphi}_1^+(w \gamma^{-\frac{1}{2}}) \widehat{\Psi}_{\Lambda_{\sigma_1} 2l+1}^{\Lambda_{\sigma_2} V}(z) \\ & \frac{q^{2l-2} z \omega}{w} (\widehat{\Psi}_{\Lambda_{\sigma_1} 2l+1}^{\Lambda_{\sigma_2} V}(z) \widehat{\varphi}_1^-(w \gamma^{\frac{1}{2}}) - \widehat{\varphi}_1^-(w \gamma^{\frac{1}{2}}) \widehat{\Psi}_{\Lambda_{\sigma_1} 2l+1}^{\Lambda_{\sigma_2} V}(z)) \\ &= \widehat{\Psi}_{\Lambda_{\sigma_1} 2l+1}^{\Lambda_{\sigma_2} V}(z) \widehat{\varphi}_1^-(w \gamma^{\frac{1}{2}}) - q^{-2} \widehat{\varphi}_1^-(w \gamma^{\frac{1}{2}}) \widehat{\Psi}_{\Lambda_{\sigma_1} 2l+1}^{\Lambda_{\sigma_2} V}(z) \end{aligned}$$

where  $\widehat{\varphi}_i^\pm(w) = q^{\mp h_i} \varphi_i^\pm(w)$ . The above two formulae hold to any powers of  $w$ . Thus we obtain

$$\begin{aligned} [a_1(n), \widehat{\Psi}_{\Lambda_{\sigma_1} 2l+1}^{\Lambda_{\sigma_2} V}(z)] &= \frac{[n]}{n} (q^{2l+\frac{1}{2}} z \omega)^n \widehat{\Psi}_{\Lambda_{\sigma_1} 2l+1}^{\Lambda_{\sigma_2} V}(z) \\ [a_1(-n), \widehat{\Psi}_{\Lambda_{\sigma_1} 2l+1}^{\Lambda_{\sigma_2} V}(z)] &= -\frac{[n]}{n} (q^{2l-\frac{1}{2}} z \omega)^{-n} \widehat{\Psi}_{\Lambda_{\sigma_1} 2l+1}^{\Lambda_{\sigma_2} V}(z) \end{aligned}$$

where  $n \geq 0$ . Meanwhile from the intertwining relation, we also get

$$[e_i, \widehat{\Psi}_{\Lambda_{\sigma_1} 2l+1}^{\Lambda_{\sigma_2} V}(z)] = 0, \quad i \neq 0$$

$$[f_i, \widehat{\Psi}_{\Lambda_{\sigma_1} 2l+1}^{\Lambda_{\sigma_2} V}(z)] = 0, \quad i \neq 1$$

Commuting the above two identities with  $a_1(\pm n)$ . We can easily prove

$$[x_2^+(w), \widehat{\Psi}_{\Lambda_{\sigma_1} 2l+1}^{\Lambda_{\sigma_2} V}(z)] = 0$$

$$[x_2^-(w), \widehat{\Psi}_{\Lambda_{\sigma_1} 2l+1}^{\Lambda_{\sigma_2} V}(z)] = 0$$

So we obtain

$$[a_2(n), \widehat{\Psi}_{\Lambda_{\sigma_1} 2l+1}^{\Lambda_{\sigma_2} V}(z)] = 0$$

Then repeat above procedures replacing  $a_1(\pm n)$  by  $a_2(\pm n)$ , and so on, we get

$$[x_i^+(w), \widehat{\Psi}_{\Lambda_{\sigma_1} 2l+1}^{\Lambda_{\sigma_2} V}(z)] = 0$$

$$[x_i^-(w), \widehat{\Psi}_{\Lambda_{\sigma_1} 2l+1}^{\Lambda_{\sigma_2} V}(z)] = 0, \quad i \neq 1$$

We can conclude that

$$[\Psi(w), \widehat{\Psi}_{\Lambda_{\sigma_1} 2l+1}^{\Lambda_{\sigma_2} V}(z)] = 0$$

$$[a_i(\pm n), \widehat{\Psi}_{\Lambda_{\sigma_1} 2l+1}^{\Lambda_{\sigma_2} V}(z)] = 0, \quad i \neq 1$$

Put

$$a_1^*(k) = \sum_{n=1}^{l-1} \frac{([(l-n)k] - [(l-n-1)k])}{([(lk] - [(l-1)k])[k])} a_n(k) + \frac{[k]}{([(lk] - [(l-1)k])[2k])} a_l(k)$$

we get

$$[a_i(k), a_1^*(-k)] = \delta_{i,1} \frac{[k]}{k}, \quad k > 0$$

$$[a_i(-k), a_1^*(k)] = -\delta_{i,1} \frac{[k]}{k}, \quad k < 0$$

From the following relations:

$$[\Psi(w), \widehat{\Psi}_{\Lambda_{\sigma_1} 2l+1}^{\Lambda_{\sigma_2} V}(z)] = 0$$

$$[a_i(\pm n), \widehat{\Psi}_{\Lambda_{\sigma_1} 2l+1}^{\Lambda_{\sigma_2} V}(z)] = 0, \quad i \neq 1$$

$$[a_1(n), \widehat{\Psi}_{\Lambda_{\sigma_1} 2l+1}^{\Lambda_{\sigma_2} V}(z)] = \frac{[n]}{n} (q^{2l+\frac{1}{2}} z \omega)^n \widehat{\Psi}_{\Lambda_{\sigma_1} 2l+1}^{\Lambda_{\sigma_2} V}(z), \quad n \in \mathbf{Z}_{>0}$$

$$[a_1(-n), \widehat{\Psi}_{\Lambda_{\sigma_1} 2l+1}^{\Lambda_{\sigma_2} V}(z)] = -\frac{[n]}{n} (q^{2l-\frac{1}{2}} z \omega)^{-n} \widehat{\Psi}_{\Lambda_{\sigma_1} 2l+1}^{\Lambda_{\sigma_2} V}(z), \quad n \in \mathbf{Z}_{>0}$$

$$[x_i^+(w), \widehat{\Psi}_{\Lambda_{\sigma_1} 2l+1}^{\Lambda_{\sigma_2} V}(z)] = 0$$

$$q^{h_1} \widehat{\Psi}_{\Lambda_{\sigma_1} 2l+1}^{\Lambda_{\sigma_2} V}(z) q^{-h_1} = q \widehat{\Psi}_{\Lambda_{\sigma_1} 2l+1}^{\Lambda_{\sigma_2} V}(z)$$

$$q^{h_i} \widehat{\Psi}_{\Lambda_{\sigma_1} 2l+1}^{\Lambda_{\sigma_2} V}(z) q^{-h_i} = \widehat{\Psi}_{\Lambda_{\sigma_1} 2l+1}^{\Lambda_{\sigma_2} V}(z), \quad i \geq 2$$

$$q^d \widehat{\Psi}_{\Lambda_{\sigma_1} 2l+1}^{\Lambda_{\sigma_2} V}(z) q^{-d} = \widehat{\Psi}_{\Lambda_{\sigma_1} 2l+1}^{\Lambda_{\sigma_2} V}(q^{-1} z)$$

we obtain the type-I vertex operators

$$\widehat{\Psi}_{\Lambda_i 2l+1}^{\Lambda_{1-i} V}(z) = \omega^{-i} \exp \left[ \sum_{n=1}^{\infty} (q^{2l+\frac{1}{2}} z \omega)^n a_1^*(-n) \right] \exp \left[ - \sum_{n=1}^{\infty} (q^{2l-\frac{1}{2}} z \omega)^{-n} a_1^*(n) \right] e^{\lambda_1} (q^{2l} z \omega)^{\partial_{\lambda_1} + i} (-1)^{2\partial_{\lambda_1}}$$

where  $i = 0, 1$

$$\begin{aligned}
\widehat{\Psi}_{\Lambda_l 2l+1}^{\Lambda_l V}(z) &= \omega^{-\frac{1}{2}} \exp\left[\sum_{n=1}^{\infty} (q^{2l+\frac{1}{2}} z \omega)^n a_1^*(-n)\right] \exp\left[-\sum_{n=1}^{\infty} (q^{2l-\frac{1}{2}} z \omega)^{-n} a_1^*(n)\right] e^{\lambda_1} (q^{2l} z \omega)^{\partial_{\lambda_1}} (-1)^{2\partial_{\lambda_l}} \\
\widehat{\Psi}_{\Lambda_{\sigma_1} 2l+1-n}^{\Lambda_{\sigma_2} V}(z) &= \widehat{\Psi}_{\Lambda_{\sigma_1} 2l+2-n}^{\Lambda_{\sigma_2} V}(z) f_n - q f_n \widehat{\Psi}_{\Lambda_{\sigma_1} 2l+2-n}^{\Lambda_{\sigma_2} V}(z), \quad n < l \\
\widehat{\Psi}_{\Lambda_{\sigma_1} n}^{\Lambda_{\sigma_2} V}(z) &= \widehat{\Psi}_{\Lambda_{\sigma_1} n+1}^{\Lambda_{\sigma_2} V}(z) f_n - q f_n \widehat{\Psi}_{\Lambda_{\sigma_1} n+1}^{\Lambda_{\sigma_2} V}(z), \quad n < l \\
\widehat{\Psi}_{\Lambda_{\sigma_1} l+1}^{\Lambda_{\sigma_2} V}(z) &= \omega^{\frac{1}{2}} (\widehat{\Psi}_{\Lambda_{\sigma_1} l+2}^{\Lambda_{\sigma_2} V}(z) f_l - q f_l \widehat{\Psi}_{\Lambda_{\sigma_1} l+2}^{\Lambda_{\sigma_2} V}(z)) \\
\widehat{\Psi}_{\Lambda_{\sigma_1} l}^{\Lambda_{\sigma_2} V}(z) &= \omega^{\frac{1}{2}} (\widehat{\Psi}_{\Lambda_{\sigma_1} l+1}^{\Lambda_{\sigma_2} V}(z) f_l - f_l \widehat{\Psi}_{\Lambda_{\sigma_1} l+1}^{\Lambda_{\sigma_2} V}(z))
\end{aligned}$$

where we have used the normalization condition for  $i = 0, 1, l$ .

### 3.4 Vertex Operator of Type-II

For the Vertex Operator of type-II, the analogous commutation relations can be got. We have

$$\begin{aligned}
[\Psi(W), \widehat{\Phi}_{\Lambda_{\sigma_2} 1}^{V \Lambda_{\sigma_1}}(z)] &= 0 \\
[a_1(n), \widehat{\Phi}_{\Lambda_{\sigma_2} 1}^{V \Lambda_{\sigma_1}}(z)] &= -\frac{[n]}{n} (q^{\frac{1}{2}} \omega z)^n \widehat{\Phi}_{\Lambda_{\sigma_2} 1}^{V \Lambda_{\sigma_1}}(z) \\
[a_1(-n), \widehat{\Phi}_{\Lambda_{\sigma_2} 1}^{V \Lambda_{\sigma_1}}(z)] &= \frac{[n]}{n} (q^{-\frac{1}{2}} \omega z)^{-n} \widehat{\Phi}_{\Lambda_{\sigma_2} 1}^{V \Lambda_{\sigma_1}}(z) \\
[a_i(\pm n), \widehat{\Phi}_{\Lambda_{\sigma_2} 1}^{V \Lambda_{\sigma_1}}(z)] &= 0, \quad i \neq 1 \\
[x_i^-, \widehat{\Phi}_{\Lambda_{\sigma_2} 1}^{V \Lambda_{\sigma_1}}(z)] &= 0 \\
q^{h_1} \widehat{\Phi}_{\Lambda_{\sigma_2} 1}^{V \Lambda_{\sigma_1}}(z) q^{-h_1} &= q^{-1} \widehat{\Phi}_{\Lambda_1 1}^{V \Lambda_{\sigma_1}}(z) \\
q^{h_i} \widehat{\Phi}_{\Lambda_{\sigma_2} 1}^{V \Lambda_{\sigma_1}}(z) q^{-h_i} &= \widehat{\Phi}_{\Lambda_{\sigma_2} 1}^{V \Lambda_{\sigma_1}}(z), \quad i \geq 2 \\
q^{-d} \widehat{\Phi}_{\Lambda_{\sigma_2} 1}^{V \Lambda_{\sigma_1}}(z) q^d &= \widehat{\Phi}_{\Lambda_{\sigma_2} 1}^{V \Lambda_{\sigma_1}}(qz)
\end{aligned}$$

So we can get the type-II vertex operators

$$\widehat{\Phi}_{\Lambda_1 1}^{V \Lambda_{1-i}}(z) = (\omega q^{2l-1})^{i-1} \exp\left[-\sum_{n=1}^{\infty} (q^{\frac{1}{2}} z \omega)^n a_1^*(-n)\right] \exp\left[\sum_{n=1}^{\infty} (q^{-\frac{1}{2}} z \omega)^{-n} a_1^*(n)\right] e^{-\lambda_1} (z \omega)^{-\partial_{\lambda_1} + i} (-1)^{2\partial_{\lambda_l}}$$

where  $i = 0, 1$

$$\widehat{\Phi}_{\Lambda_l 1}^{V \Lambda_l}(z) = \omega^{-\frac{1}{2}} (-q)^{-l} \exp\left[-\sum_{n=1}^{\infty} (q^{\frac{1}{2}} z \omega)^n a_1^*(-n)\right] \exp\left[\sum_{n=1}^{\infty} (q^{-\frac{1}{2}} z \omega)^{-n} a_1^*(n)\right] e^{-\lambda_1} (z \omega)^{-\partial_{\lambda_1}} (-1)^{2\partial_{\lambda_l}}$$

$$\begin{aligned}
\widehat{\Phi}_{\Lambda_{\sigma_2}n+1}^{V\Lambda_{\sigma_1}}(z) &= \widehat{\Phi}_{\Lambda_{\sigma_2}n}^{V\Lambda_{\sigma_1}}(z)e_n - qe_n\widehat{\Phi}_{\Lambda_i n}^{V\Lambda_j}(z), \quad n < l \\
\widehat{\Phi}_{\Lambda_{\sigma_2}2l+2-n}^{V\Lambda_{\sigma_1}}(z) &= \widehat{\Phi}_{\Lambda_{\sigma_2}2l+1-n}^{V\Lambda_{\sigma_1}}(z)e_n - qe_n\widehat{\Phi}_{\Lambda_i 2l+1-n}^{V\Lambda_j}(z), \quad n < l \\
\widehat{\Phi}_{\Lambda_{\sigma_2}l+2}^{V\Lambda_{\sigma_1}}(z) &= \omega^{\frac{1}{2}}(\widehat{\Phi}_{\Lambda_{\sigma_2}l+1}^{V\Lambda_{\sigma_1}}(z)e_l - e_l\widehat{\Phi}_{\Lambda_{\sigma_2}l+1}^{V\Lambda_{\sigma_1}}(z)) \\
\widehat{\Phi}_{\Lambda_{\sigma_2}l+1}^{V\Lambda_{\sigma_1}}(z) &= \omega^{\frac{1}{2}}(\widehat{\Phi}_{\Lambda_{\sigma_2}l}^{V\Lambda_{\sigma_1}}(z)e_l - qe_l\widehat{\Phi}_{\Lambda_{\sigma_2}l}^{V\Lambda_{\sigma_1}}(z))
\end{aligned}$$

### 3.5 Dual Vertex Operators

Define the intertwiners of the form

$$\widehat{\Psi}_{\Lambda_j V}^{*\Lambda_i}(z) : V(\Lambda_j) \otimes V_z \rightarrow V(\Lambda_i)$$

$$\widehat{\Phi}_{V\Lambda_j}^{*\Lambda_i}(z) : V_z \otimes V(\Lambda_j) \rightarrow V(\Lambda_i)$$

They are called dual vertex operators. Define their components by

$$\widehat{\Psi}_{\Lambda_j V m}^{*\Lambda_i}(z) | v \rangle = \widehat{\Psi}_{\Lambda_j V}^{*\Lambda_i}(z) (| v \rangle \otimes v_m)$$

$$\widehat{\Phi}_{V\Lambda_j m}^{*\Lambda_i}(z) | v \rangle = \widehat{\Phi}_{V\Lambda_j}^{*\Lambda_i}(z) (v_m \otimes | v \rangle)$$

We impose the normalization

$$\begin{aligned}
\langle \Lambda_1 | \widehat{\Psi}_{\Lambda_0 V 1}^{*\Lambda_1}(z) | \Lambda_0 \rangle &= 1, \quad \langle \Lambda_1 | \widehat{\Phi}_{V\Lambda_0 1}^{*\Lambda_1}(z) | \Lambda_0 \rangle = 1 \\
\langle \Lambda_0 | \widehat{\Psi}_{\Lambda_1 V 2l+1}^{*\Lambda_0}(z) | \Lambda_1 \rangle &= 1, \quad \langle \Lambda_0 | \widehat{\Phi}_{V\Lambda_1 2l+1}^{*\Lambda_0}(z) | \Lambda_1 \rangle = 1 \\
\langle \Lambda'_l | \widehat{\Psi}_{\Lambda_l V l+1}^{*\Lambda'_l}(z) | \Lambda_l \rangle &= 1, \quad \langle \Lambda'_l | \widehat{\Phi}_{V\Lambda_l l+1}^{*\Lambda'_l}(z) | \Lambda_l \rangle = 1
\end{aligned}$$

With analogue dicussion in [7] p79, we get

$$\begin{aligned}
\widehat{\Phi}_{V\Lambda_i 2l+1}^{*\Lambda_{1-i}}(z) &= q^{(2l-1)(1-i)} \widehat{\Phi}_{\Lambda_i 1}^{V\Lambda_{1-i}}(q^{2l-1}z) \\
\widehat{\Psi}_{\Lambda_i V 1}^{*\Lambda_{1-i}}(z) &= q^{(2l-1)i} \widehat{\Psi}_{\Lambda_i 2l+1}^{\Lambda_{1-i} V}(q^{-2l+1}z) \\
\widehat{\Phi}_{V\Lambda_l 2l+1}^{*\Lambda_l}(z) &= (-q)^l \widehat{\Phi}_{\Lambda_l 1}^{V\Lambda_l}(q^{2l-1}z) \\
\widehat{\Psi}_{\Lambda_l V 1}^{*\Lambda_l}(z) &= (-q)^l \widehat{\Psi}_{\Lambda_l 2l+1}^{\Lambda_l V}(q^{-2l+1}z) \\
\widehat{\Psi}_{\Lambda_{\sigma_1} 2l+2-n}^{*\Lambda_{\sigma_2} V}(z) &= f_n \widehat{\Psi}_{\Lambda_{\sigma_1} 2l+1-n}^{*\Lambda_{\sigma_2} V}(z) - q^{-1} \widehat{\Psi}_{\Lambda_{\sigma_1} 2l+1-n}^{*\Lambda_{\sigma_2} V}(z) f_n, \quad n < l
\end{aligned}$$

$$\begin{aligned}
\widehat{\Psi}_{\Lambda_{\sigma_1}n+1}^{*\Lambda_{\sigma_2}V}(z) &= f_n \widehat{\Psi}_{\Lambda_{\sigma_1}n}^{*\Lambda_{\sigma_2}V}(z) - q^{-1} \widehat{\Psi}_{\Lambda_{\sigma_1}n}^{*\Lambda_{\sigma_2}V}(z) f_n, \quad n < l \\
\widehat{\Psi}_{\Lambda_{\sigma_1}l+2}^{*\Lambda_{\sigma_2}V}(z) &= \omega^{\frac{1}{2}} (f_l \widehat{\Psi}_{\Lambda_{\sigma_1}l+1}^{*\Lambda_{\sigma_2}V}(z) - \widehat{\Psi}_{\Lambda_{\sigma_1}l+1}^{*\Lambda_{\sigma_2}V}(z) f_l) \\
\widehat{\Psi}_{\Lambda_{\sigma_1}l+1}^{*\Lambda_{\sigma_2}V}(z) &= \omega^{\frac{1}{2}} (f_l \widehat{\Psi}_{\Lambda_{\sigma_1}l}^{*\Lambda_{\sigma_2}V}(z) - q^{-1} \widehat{\Psi}_{\Lambda_{\sigma_1}l}^{*\Lambda_{\sigma_2}V}(z) f_l) \\
\widehat{\Phi}_{\Lambda_{\sigma_2}n}^{*V\Lambda_{\sigma_1}}(z) &= e_n \widehat{\Phi}_{\Lambda_{\sigma_2}n+1}^{*V\Lambda_{\sigma_1}}(z) - q^{-1} \widehat{\Phi}_{\Lambda_{\sigma_2}n+1}^{*V\Lambda_j}(z) e_n, \quad n < l \\
\widehat{\Phi}_{\Lambda_{\sigma_2}2l+1-n}^{*V\Lambda_{\sigma_1}}(z) &= e_n \widehat{\Phi}_{\Lambda_{\sigma_2}2l+2-n}^{*V\Lambda_{\sigma_1}}(z) - q^{-1} \widehat{\Phi}_{\Lambda_{\sigma_2}2l+2-n}^{*V\Lambda_j}(z) e_n, \quad n < l \\
\widehat{\Phi}_{\Lambda_{\sigma_2}l+1}^{*V\Lambda_{\sigma_1}}(z) &= \omega^{\frac{1}{2}} (e_l \widehat{\Phi}_{\Lambda_{\sigma_2}l+2}^{*V\Lambda_{\sigma_1}}(z) - q^{-1} \widehat{\Phi}_{\Lambda_{\sigma_2}l+2}^{*V\Lambda_{\sigma_1}}(z) e_l) \\
\widehat{\Phi}_{\Lambda_{\sigma_2}l}^{*V\Lambda_{\sigma_1}}(z) &= \omega^{\frac{1}{2}} (e_l \widehat{\Phi}_{\Lambda_{\sigma_2}l+1}^{*V\Lambda_{\sigma_1}}(z) - \widehat{\Phi}_{\Lambda_{\sigma_2}l+1}^{*V\Lambda_{\sigma_1}}(z) e_l)
\end{aligned}$$

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